# Normal Forms for Modal Logics

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# Motivation

- Normal modal logics have been used in computer science to represent complex situations, e.g. multi-agent and distributed systems;
- Verification of properties of those systems may require the combination of proof methods;
- Provide a set of tools, based on clausal resolution, to tackle these problems.

# Axioms

$$\mathbf{K}: \quad \Box(\varphi \Rightarrow \psi) \Rightarrow (\Box\varphi \Rightarrow \Box\psi)$$

$$\mathbf{T}: \quad \Box\varphi \Rightarrow \varphi$$

$$\mathbf{D}: \quad \Box\varphi \Rightarrow \Diamond\varphi$$

$$\mathbf{4}: \quad \Box\varphi \Rightarrow \Box\Box\varphi$$

$$\mathbf{5}: \quad \Diamond\varphi \Rightarrow \Box\Diamond\varphi$$

$$\mathbf{B}: \quad \Diamond\Box\varphi \Rightarrow \varphi$$

Table 1: Usual Axioms for Normal Modal Logics

# Normal Modal Logics

There are 24 possible combinations, but because

**T** implies **D**

**B** and **5** implies **4**

**T** and **5** implies **B**

**4**, **B**, and **D** implies **T**

there are fifteen modal systems:

$$
\begin{array}{lll}
K & & \\
T & KTB = B & K4B \\
KD & KDB & KD45 \\
KB & KD5 & KT4 = S4 \\
K4 & KD4 & KT5 = S5 \\
K5 & K45 &
\end{array}
$$

# Resolution and Correspondence Theory

## Reflexivity

$$\frac{x \Rightarrow \boxed{i}\,p}{x \Rightarrow p}$$

## Symmetry

$$\frac{\begin{array}{l} x \Rightarrow \boxed{i}\,p \\ y \Rightarrow \neg\boxed{i}\,p \end{array}}{x \Rightarrow \boxed{i}\,\neg y}$$

# Normal Logic $K_{(n)}$

Given a set of agents $\mathcal{A} = \{1, \ldots, n\}$

- **propositional symbols**: $\mathcal{P} = \{p, q, r, \ldots, p_1, q_1, r_1, \ldots\}$
- **classical connectives**: $\{\neg, \vee, \wedge, \Rightarrow\}$;
- **modal operators**: $\{\boxed{1}, \ldots, \boxed{n}\}$

**Well-formed formulae** ($\mathsf{WFF}_{\mathsf{K}_n}$) are recursively defined:

- $p \in \mathcal{P}$ is in $\mathsf{WFF}_{\mathsf{K}_n}$;
- if $\varphi$ and $\psi$ are in $\mathsf{WFF}_{\mathsf{K}_n}$, then so are:

$$\neg(\varphi), (\varphi \vee \psi), (\varphi \wedge \psi), (\varphi \Rightarrow \psi), \boxed{1}\varphi, \ldots, \boxed{n}\varphi$$

# Semantics

A Kripke structures for $n$ agents over $\mathcal{P}$ is a tuple:

$$M = \langle \mathcal{S}, \pi, \mathcal{R}_1, \ldots, \mathcal{R}_n \rangle$$

where

- where $\mathcal{S}$ is a non-empty set, with a distinguished world $s_0$;
- $\pi$ is a function $\pi(s) : \mathcal{P} \longrightarrow \{\mathsf{V}, \mathsf{F}\}$; and
- $\mathcal{R}_i \subseteq \mathcal{S} \times \mathcal{S}$ are binary relations over $S$.

# Interpretation of Formulae

Let $p \in \mathcal{P}$ and $\varphi$, $\psi \in$ $\mathsf{WFF}_{\mathsf{K}_n}$ e $M = \langle \mathcal{S}, \pi, \mathcal{K}_1, \ldots, \mathcal{K}_n \rangle$:

- $(M, s) \models$ **true**
- $(M, s) \not\models$ **false**
- $(M, s) \models p$ if, and only if, $\pi(s)(p) =$ *true*, where $p \in \mathcal{P}$
- $(M, s) \models \neg\varphi$ if, and only if, $(M, s) \not\models \varphi$
- $(M, s) \models (\varphi \land \psi)$ if, and only if, $(M, s) \models \varphi$ and $(M, s) \models \psi$
- $(M, s) \models (\varphi \lor \psi)$ if, and only if, $(M, s) \models \varphi$ or $(M, s) \models \psi$
- $(M, s) \models (\varphi \Rightarrow \psi)$ if, and only if, $(M, s) \models \neg\varphi$ or $(M, s) \models \psi$
- $(M, s) \models (\varphi \Leftrightarrow \psi)$ if, and only if, $(M, s) \models (\varphi \Rightarrow \psi)$ and $(M, s) \models (\psi \Rightarrow \varphi)$
- $(M, s) \models \boxed{i}\,\varphi$ if, and only if, for all $t$, such that $(s, t) \in \mathcal{R}_i$, $(M, t) \models \varphi$.

# Normal Form for $K_{(n)}$

$$\Box^* \bigwedge_i A_i$$

where

- Initial clause $$\textbf{start} \Rightarrow \bigvee_{b=1}^{r} l_b$$
- Literal clause $$\textbf{true} \Rightarrow \bigvee_{b=1}^{r} l_b$$
- $\boxed{i}$-clause $$l \Rightarrow m_i$$

where $l$ and any $l_b$ are literals and $m_i$ is a modal literal containing a $\boxed{i}$ or a $\neg\boxed{i}$ operator.

# Transformation

We introduce the nullary connective **start**, where $(M, s) \models$ **start** if, and only if, $s = s_0$, and apply the transformation rules by anchoring the translation to the initial world:

$$\tau_0(\varphi) = \Box^*(\textbf{start} \Rightarrow f) \wedge \tau_1(\Box^*(f \Rightarrow \varphi))$$

and doing classical style rewriting for most of the classical operators:

$$\begin{array}{rcl}
\tau_1(\Box^*(x \Rightarrow \neg\neg A)) & = & \tau_1(\Box^*(x \Rightarrow A)) \\
\tau_1(\Box^*(x \Rightarrow (A \wedge B))) & = & \tau_1(\Box^*(x \Rightarrow A)) \wedge \tau_1(\Box^*(x \Rightarrow B)) \\
\tau_1(\Box^*(x \Rightarrow (A \Rightarrow B))) & = & \tau_1(\Box^*(x \Rightarrow \neg A \vee B)) \\
\tau_1(\Box^*(x \Rightarrow \neg(A \wedge B))) & = & \tau_1(\Box^*(x \Rightarrow \neg A \vee \neg B)) \\
\tau_1(\Box^*(x \Rightarrow \neg(A \Rightarrow B))) & = & \tau_1(\Box^*(x \Rightarrow A)) \wedge \tau_1(\Box^*(x \Rightarrow \neg B)) \\
\tau_1(\Box^*(x \Rightarrow \neg(A \vee B))) & = & \tau_1(\Box^*(x \Rightarrow \neg A)) \wedge \tau_1(\Box^*(x \Rightarrow \neg B))
\end{array}$$

# Transformation – Continued

We rename complex formulae in double implications:

$$\begin{aligned}
\tau_1(\Box^*(x \Rightarrow (A \Leftrightarrow B))) &= \tau_1(\Box^*(x \Rightarrow \neg y \vee z)) \wedge \tau_1(\Box^*(x \Rightarrow \neg z \vee y)) \wedge \\
&\quad \tau_1(\Box^*(y \Rightarrow A)) \wedge \tau_1(\Box^*(\neg y \Rightarrow \neg A)) \wedge \\
&\quad \tau_1(\Box^*(z \Rightarrow B)) \wedge \tau_1(\Box^*(\neg z \Rightarrow \neg B)) \\
\tau_1(\Box^*(x \Rightarrow \neg(A \Leftrightarrow B))) &= \tau_1(\Box^*(x \Rightarrow ((y \wedge \neg z) \vee (z \wedge \neg y)))) \wedge \\
&\quad \tau_1(\Box^*(y \Rightarrow A)) \wedge \tau_1(\Box^*(\neg y \Rightarrow \neg A)) \wedge \\
&\quad \tau_1(\Box^*(z \Rightarrow B)) \wedge \tau_1(\Box^*(\neg z \Rightarrow \neg B))
\end{aligned}$$

We also rename complex formulae in the scope of a modal operator:

$$\begin{aligned}
\tau_1(\Box^*(x \Rightarrow \boxed{i} A)) &= \tau_1(\Box^*(x \Rightarrow \boxed{i} y)) \wedge \tau_1(\Box^*(y \Rightarrow A)) \\
\tau_1(\Box^*(x \Rightarrow \neg \boxed{i} A)) &= \tau_1(\Box^*(x \Rightarrow \neg \boxed{i} \neg y)) \wedge \tau_1(\Box^*(y \Rightarrow \neg A))
\end{aligned}$$

# Transformation – Continued

Implications are rewritten as disjunctions:

$$\tau_1(\square^*(x \Rightarrow D \vee (D' \Rightarrow D''))) = \tau_1(\square^*(x \Rightarrow D \vee \neg D' \vee D''))$$

and disjunctions are then, finally, rewritten in the right form:

$$\tau_1(\square^*(x \Rightarrow D \vee A)) = \tau_1(\square^*(x \Rightarrow D \vee y)) \wedge \tau_1(\square^*(y \Rightarrow A))$$

$$\tau_1(\square^*(x \Rightarrow D)) = \begin{cases} \square^*(\textbf{true} \Rightarrow \neg x \vee D) & \text{if } D \text{ is a disjunction of literals} \\ \square^*(x \Rightarrow D) & \text{if } D \text{ is a modal literal} \end{cases}$$

# Example

$$\boxed{i}\,(a \wedge \boxed{i}\,(b \wedge \boxed{i}\,c))$$

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# Example

$$\boxed{i}(a \wedge \boxed{i}(b \wedge \boxed{i}c))$$

$$(x \Rightarrow \boxed{i}y) \wedge (y \Rightarrow (a \wedge \boxed{i}(b \wedge \boxed{i}c)))$$

$$(y \Rightarrow a) \wedge (y \Rightarrow \boxed{i}(b \wedge \boxed{i}c))$$

$$(y \Rightarrow \boxed{i}z) \wedge (z \Rightarrow b \wedge \boxed{i}c)$$

# Example

$$\boxed{i}(a \wedge \boxed{i}(b \wedge \boxed{i}c))$$

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$$(y \Rightarrow a) \wedge (y \Rightarrow \boxed{i}(b \wedge \boxed{i}c))$$

$$(y \Rightarrow \boxed{i}z) \wedge (z \Rightarrow b \wedge \boxed{i}c)$$

Giving:

| | | | |
|---|---|---|---|
| 1. | **start** | $\Rightarrow$ | $x$ |
| 2. | $x$ | $\Rightarrow$ | $\boxed{i}y$ |
| 3. | **true** | $\Rightarrow$ | $\neg y \vee a$ |
| 4. | $y$ | $\Rightarrow$ | $\boxed{i}z$ |
| 5. | **true** | $\Rightarrow$ | $\neg z \vee b$ |
| 6. | $z$ | $\Rightarrow$ | $\boxed{i}c$ |

## Note on the new normal form

There is little difference between this normal form and the one previously used in the combination of epistemic and temporal logics. For instance, where previously we had:

$$x \Rightarrow l_1 \vee \ldots \vee l_p \vee \boxed{i}\, m_1 \vee \ldots \vee \boxed{i}\, m_q \vee \neg \boxed{i}\, n_1 \vee \ldots \vee \neg \boxed{i}\, n_r$$

we now have

$$x \Rightarrow l_1 \vee \ldots \vee l_p \vee m_1' \vee \ldots \vee m_q' \vee n_1' \vee \ldots \vee n_r'$$

and

$$\begin{array}{rcl} m_1' & \Rightarrow & \boxed{i}\, m_1 \\ & \vdots & \\ m_q' & \Rightarrow & \boxed{i}\, m_q \\ n_1' & \Rightarrow & \neg \boxed{i}\, n_1 \\ & \vdots & \\ n_r' & \Rightarrow & \neg \boxed{i}\, n_r \end{array}$$

# Results

The transformation into the normal form preserves satisfiability.

- Let $\varphi$ be a formula in $\mathsf{K}_{(n)}$. If $\models \tau_0(\varphi)$, then $\models \varphi$.
- Let $\varphi$ be a formula in $\mathsf{K}_{(n)}$. If $\models \varphi$, then $\models \tau_0(\varphi)$.

# Anti-Prenexing

Pushing modal operators as far as we can, accordingly to the follow equivalences:

$$
\begin{array}{lrcl}
1. & \boxed{i}(\varphi \wedge \psi) & \Leftrightarrow & (\boxed{i}\varphi \wedge \boxed{i}\psi) \\
2. & \boxed{i}\neg(\varphi \Rightarrow \psi) & \Leftrightarrow & (\boxed{i}\varphi \wedge \boxed{i}\neg\psi) \\
3. & \boxed{i}\neg(\varphi \vee \psi) & \Leftrightarrow & (\boxed{i}\neg\varphi \wedge \boxed{i}\neg\psi) \\
4. & \neg\boxed{i}\neg(\varphi \Rightarrow \psi) & \Leftrightarrow & (\boxed{i}\varphi \Rightarrow \neg\boxed{i}\neg\psi) \\
5. & \neg\boxed{i}\neg(\varphi \vee \psi) & \Leftrightarrow & (\neg\boxed{i}\neg\varphi \vee \neg\boxed{i}\neg\psi) \\
6. & \neg\boxed{i}(\varphi \wedge \psi) & \Leftrightarrow & (\neg\boxed{i}\varphi \vee \neg\boxed{i}\psi)
\end{array}
$$

# Examples

$$\boxed{i}(a \wedge \boxed{i}(b \wedge \boxed{i}c))$$

is equivalent to

$$\boxed{i}a \wedge \boxed{i}\boxed{i}b \wedge \boxed{i}\boxed{i}\boxed{i}c$$

and

$$\neg\boxed{i}\neg(a \vee \neg\boxed{i}\neg(b \vee \neg\boxed{i}\neg c))$$

is equivalent to

$$\neg\boxed{i}\neg a \vee \neg\boxed{i}\neg\neg\boxed{i}\neg b \vee \neg\boxed{i}\neg\neg\boxed{i}\neg\neg\boxed{i}\neg c$$

# Is this any good?

In some cases, we can get smaller sets of clauses, when the formula is firstly transformed into its anti-prenex form:

$$\boxed{i}(a \wedge b)$$

| SNF only | AP + SNF |
|---|---|
| $\boxed{i}(a \wedge b)$ | $\boxed{i}a \wedge \boxed{i}b$ |

| SNF only | | | | AP + SNF | | | |
|---|---|---|---|---|---|---|---|
| 1. | **start** | $\Rightarrow$ | $x$ | 1. | **start** | $\Rightarrow$ | $x$ |
| 2. | $x$ | $\Rightarrow$ | $\boxed{i}y$ | 2. | $x$ | $\Rightarrow$ | $\boxed{i}a$ |
| 3. | **true** | $\Rightarrow$ | $\neg y \vee a$ | 3. | $x$ | $\Rightarrow$ | $\boxed{i}b$ |
| 4. | $y$ | $\Rightarrow$ | $\boxed{i}b$ | | | | |

# Is this always the case?

$$\boxed{i}(a \wedge \boxed{i}b)$$

| SNF only | AP + SNF |
|---|---|
| $\boxed{i}(a \wedge \boxed{i}b)$ | $\boxed{i}a \wedge \boxed{i}\boxed{i}b$ |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1. | **start** | $\Rightarrow$ | $x$ | 1. | **start** | $\Rightarrow$ | $x$ |
| 2. | $x$ | $\Rightarrow$ | $\boxed{i}y$ | 2. | $x$ | $\Rightarrow$ | $\boxed{i}a$ |
| 3. | **true** | $\Rightarrow$ | $\neg y \vee a$ | 3. | $x$ | $\Rightarrow$ | $\boxed{i}y$ |
| 4. | **true** | $\Rightarrow$ | $\neg y \vee b$ | 4. | $y$ | $\Rightarrow$ | $\boxed{i}b$ |

# Nope! This is not always the case!

$$\boxed{i}(a \wedge \boxed{i}(b \wedge \boxed{i}c))$$

| SNF only | AP + SNF |
|---|---|
| $\boxed{i}(a \wedge \boxed{i}(b \wedge \boxed{i}c))$ | $\boxed{i}a \wedge \boxed{i}\boxed{i}b \wedge \boxed{i}\boxed{i}\boxed{i}c$ |

| | SNF only | | | | AP + SNF | | |
|---|---|---|---|---|---|---|---|
| 1. | **start** | $\Rightarrow$ | $x$ | 1. | **start** | $\Rightarrow$ | $x$ |
| 2. | $x$ | $\Rightarrow$ | $\boxed{i}y$ | 2. | $x$ | $\Rightarrow$ | $\boxed{i}a$ |
| 3. | **true** | $\Rightarrow$ | $\neg y \vee a$ | 3. | $x$ | $\Rightarrow$ | $\boxed{i}y$ |
| 4. | $y$ | $\Rightarrow$ | $\boxed{i}z$ | 4. | $y$ | $\Rightarrow$ | $\boxed{i}b$ |
| 5. | **true** | $\Rightarrow$ | $\neg z \vee b$ | 5. | $x$ | $\Rightarrow$ | $\boxed{i}z$ |
| 6. | $z$ | $\Rightarrow$ | $\boxed{i}c$ | 6. | $z$ | $\Rightarrow$ | $\boxed{i}w$ |
| | | | | 7. | $w$ | $\Rightarrow$ | $\boxed{i}c$ |

# However...

For few of the normal modal logics considered, simplification steps can be made, because the following equivalences hold:

| Equivalences | Valid in | Not Valid in |
|---|---|---|
| $\boxed{i}\,\boxed{i}\,\varphi \Leftrightarrow \boxed{i}\,\varphi$ | K45$_{(n)}$KD45$_{(n)}$, S4$_{(n)}$, S5$_{(n)}$ | $*$ |
| $\boxed{i}\,\neg\boxed{i}\,\varphi \Leftrightarrow \neg\boxed{i}\,\varphi$ | KD45$_{(n)}$, S5$_{(n)}$ | K45$_{(n)}$, S4$_{(n)}$, $*$ |
| $\neg\boxed{i}\,\neg\boxed{i}\,\varphi \Leftrightarrow \boxed{i}\,\varphi$ | KD45$_{(n)}$, S5$_{(n)}$ | K45$_{(n)}$, S4$_{(n)}$, $*$ |
| $\neg\boxed{i}\,\boxed{i}\,\varphi \Leftrightarrow \neg\boxed{i}\,\varphi$ | K45$_{(n)}$KD45$_{(n)}$, S4$_{(n)}$, S5$_{(n)}$ | $*$ |

where * is any of the other normal modal logics.

# Anti-Prenex with Simplification

$$\boxed{i}(a \wedge \boxed{i}(b \wedge \boxed{i}c))$$

| SNF only | | | | AP + Simp + SNF | | | |
|---|---|---|---|---|---|---|---|
| $\boxed{i}(a \wedge \boxed{i}(b \wedge \boxed{i}c))$ | | | | $\boxed{i}a \wedge \boxed{i}b \wedge \boxed{i}c$ | | | |
| 1. | **start** | $\Rightarrow$ | $x$ | 1. | **start** | $\Rightarrow$ | $x$ |
| 2. | $x$ | $\Rightarrow$ | $\boxed{i}y$ | 2. | $x$ | $\Rightarrow$ | $\boxed{i}a$ |
| 3. | **true** | $\Rightarrow$ | $\neg y \vee a$ | 3. | $x$ | $\Rightarrow$ | $\boxed{i}b$ |
| 4. | $y$ | $\Rightarrow$ | $\boxed{i}z$ | 4. | $x$ | $\Rightarrow$ | $\boxed{i}c$ |
| 5. | **true** | $\Rightarrow$ | $\neg z \vee b$ | | | | |
| 6. | $z$ | $\Rightarrow$ | $\boxed{i}c$ | | | | |

## Is this always good?

- All simplications rules can only be applied to two normal modal logics;
- Also, a good result depends on the structure of the formula;

  That is, we cannot prove that the size of the formula obtained by anti-prenexing together with simplification is better than SNF alone.

- Experimental results, simplifications in S5$_{(n)}$:
  - SNF alone is better than combined with anti-prenexing without simplification;
  - Anti-prenexing with simplification gives better results than SNF alone in most of the cases.

# Prenex

Anti-prenexing without simplification can generate bigger formulae, but using the same equivalences as before, that is:

$$
\begin{array}{lrcl}
1. & \boxed{i}(\varphi \wedge \psi) & \Leftrightarrow & (\boxed{i}\varphi \wedge \boxed{i}\psi) \\
2. & \boxed{i}\neg(\varphi \Rightarrow \psi) & \Leftrightarrow & (\boxed{i}\varphi \wedge \boxed{i}\neg\psi) \\
3. & \boxed{i}\neg(\varphi \vee \psi) & \Leftrightarrow & (\boxed{i}\neg\varphi \wedge \boxed{i}\neg\psi) \\
4. & \neg\boxed{i}\neg(\varphi \Rightarrow \psi) & \Leftrightarrow & (\boxed{i}\varphi \Rightarrow \neg\boxed{i}\neg\psi) \\
5. & \neg\boxed{i}\neg(\varphi \vee \psi) & \Leftrightarrow & (\neg\boxed{i}\neg\varphi \vee \neg\boxed{i}\neg\psi) \\
6. & \neg\boxed{i}(\varphi \wedge \psi) & \Leftrightarrow & (\neg\boxed{i}\varphi \vee \neg\boxed{i}\psi)
\end{array}
$$

we could obtain formulae with a smaller number of modal operators. The drawbacks are the same, however, as the anti-prenexing case.

# Combining all together

| | |
|---|---|
| $Formula$ | $\neg\boxed{i}(a \lor \neg\boxed{i}(b \lor \neg\boxed{i}\neg c))$ |
| | $\downarrow$ |
| $AP$ | $\neg\boxed{i}\neg a \lor \neg\boxed{i}\neg\neg\boxed{i}\neg b \lor \neg\boxed{i}\neg\neg\boxed{i}\neg\neg\boxed{i}\neg c$ |
| | $\downarrow$ |
| $SP + Simp$ | $\neg\boxed{i}\neg a \lor \neg\boxed{i}\neg b \lor \neg\boxed{i}\neg c$ |
| | $\downarrow$ |
| $Prenex$ | $\neg\boxed{i}\neg(a \lor b \lor c)$ |
| | $\downarrow$ |
| $SNF$ | 1. **start** $\Rightarrow x$ <br> 2. **true** $\Rightarrow \neg\boxed{i}\neg y$ <br> 2. **true** $\Rightarrow a \lor b \lor c$ |

# Conclusions

- A normal form for normal modal logics;

- The use of anti-prenex and prenex in the transformation of a formula;

- Developing metrics for when to apply these techniques;

- Future: simplification and subsumption for those logics;

- Future: the resolution rules based on the correspondence theory and the development of a method for generating models for such logics.